Engineering Notes

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A Linear Algorithm for Determining Relative Orbital State Using Angle Data

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THE linearized free-fall dynamics of a body slightly perturbed from a circular reference orbit, expressed in Cartesian local vertical coordinates of a body in the reference orbit, are[1]

$$\frac{d}{d\tau}\begin{bmatrix} x \\ z \\ x' \\ z' \\ y \\ y' \end{bmatrix} = \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & -2 & \multicolumn{2}{c}{0} \\ 0 & 3 & 2 & 0 & & \\ \hline \multicolumn{4}{c|}{0} & 0 & 1 \\ & & & & -1 & 0 \end{array}\right]\begin{bmatrix} x \\ z \\ x' \\ z' \\ y \\ y' \end{bmatrix} \tag{1}$$

where $\tau = \omega t$, and $d(\ \ )/d\tau = (\ \ )'$. The homogeneous solution of these equations can be expressed in terms of the fundamental matrix $\Phi(\tau_2,\tau_1) \equiv \Phi_{21}$, as follows:

$$[xzx'z'yy']_{\tau_2}{}^T = \Phi_{21}[xzx'z'yy']_{\tau_1}{}^T \tag{2}$$

where

$$\Phi_{21} = \left[\begin{array}{cccc|cc} 1 & 6s_{21} - 6\tau_{21} & 4s_{21} - 3\tau_{21} & -2 + 2c_{21} & & \\ 0 & 4 - 3c_{21} & 2 - 2c_{21} & s_{21} & & \\ 0 & -6 + 6c_{21} & -3 + 4c_{21} & -2s_{21} & \multicolumn{2}{c}{0} \\ 0 & 3s_{21} & 2s_{21} & c_{21} & & \\ \hline \multicolumn{4}{c|}{0} & c_{21} & s_{21} \\ & & & & -s_{21} & c_{21} \end{array}\right] = \left[\begin{array}{c|c} \Psi_{21} & 0 \\ \hline 0 & \theta_{21} \end{array}\right] \tag{3}$$

and where $\tau_{21} \equiv \tau_2 - \tau_1$, $c_{21} \equiv \cos\tau_{21}$, and $s_{21} \equiv \sin\tau_{21}$. The elements of the 4×4 Ψ_{21} matrix are denoted by Ψ_{ij21} $(i,j = 1,2,3,4)$ and the elements of the 2×2 θ_{21} matrix are denoted by θ_{ij21} $(i,j = 1,2)$. The elements of Φ_{21} are denoted by φ_{ij21} $(i,j = 1,\ldots,6)$. In the case of an impulsively thrusting vehicle (or approximately for a finite-thrust orbit provided that the burn time is much shorter than one orbital period), the solution is modified by adding the velocity increments to x', y', z' at the time of their occurrence.

It has been shown that an unknown relative state $X_0 \equiv [x_0z_0x_0'z_0'y_0y_0']^T$ can be determined from a set of measurements of the in-plane angle δ and out-of-plane angle σ (Fig. 1), provided that an impulsive thrust maneuver of known ΔV-magnitude and direction is made by the perturbed body at some time during the measurement sequence.[1] References 1 and 2 define computational procedures for computing this state manually. This Note presents algorithms for performing the same task with a digital computer.

Denote the measurements by $\delta_1,\sigma_1,\delta_2,\sigma_2,\ldots$ corresponding to times $\tau_1,\tau_2\ldots$ and states $x_1,z_1,x_1',z_1'\ldots$. Denote the burn made by the perturbed body at τ_b by $\Delta x',\Delta y',\Delta z'$, where τ_b is chosen at some time between the first and last measurement of the set, not too close to the bounds of the measurement interval to allow the change in velocity to propagate into position.

Two types of measurement-maneuver sequences are defined, the four-instant in-plane sequence, and the three-instant out-of-plane sequence. In the former, a set of four readings of δ and two of σ are taken, and the burn has only x and z components. In the latter, three pairs of δ,σ readings are taken, and the burn has a nonzero y component, the z and x components being either zero or nonzero.

The three-instant procedure is considered first. The state at time τ_i can be expressed

$$X_i = \Phi_{i0}\,X_0 + \Phi_{ib}\,V \tag{4}$$

where

$$V \equiv [000\Delta x'\Delta y'\Delta z']^T \tag{5}$$

$$\Phi_{ib} \equiv \begin{cases} [0] & \text{if } \tau_i < \tau_b \\ \Phi(\tau_i,\tau_b) & \text{if } \tau_i \geq \tau_b \end{cases} \tag{6}$$

The individual state-components at time τ_i can be expressed

$$x_i = \Phi_{xi0}\quad X_0 + \Phi_{xib}\quad V \quad (i = 1,2,3) \tag{7}$$

$$y_i = \Phi_{yi0}\quad X_0 + \Phi_{yib}\quad V \quad (i = 1,2,3) \tag{8}$$

and a similar equation for z_i, where Φ_{xi0} denotes the 1×6 matrix whose elements are the top row of Φ_{i0}, Φ_{yi0} denotes the matrix whose elements are the fifth row of Φ_{i0}; and Φ_{iz0} denotes the matrix whose elements are the second row of Φ_{i0}, and similarly for Φ_{xib}, Φ_{yib}, Φ_{zib}. Form the 3×1 matrix of x's as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \Phi_{x10} \\ \Phi_{x20} \\ \Phi_{x30} \end{bmatrix}[X_0] + \begin{bmatrix} \Phi_{x1b} \\ \Phi_{y1b} \\ \Phi_{z3b} \end{bmatrix}[V] \tag{9}$$

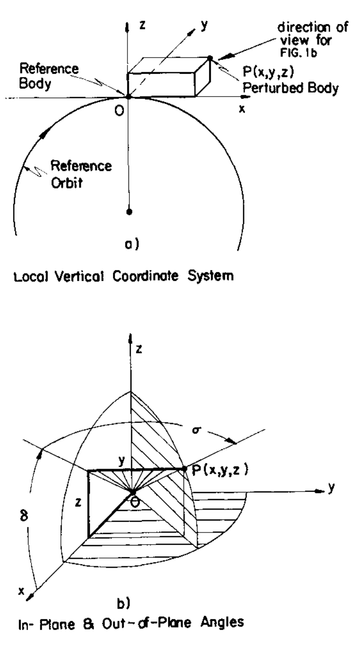

Fig. 1 Geometry of relative coordinate system showing measurement angles.

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This is abbreviated to matrix notation as

$$\mathfrak{X} = \bar{\Phi}_x X_0 + \bar{\Phi}_{xb} V \tag{10}$$

By similar steps and definitions, one obtains the vectors of the y and z components

$$\mathcal{Y} = \bar{\Phi}_y X_0 + \bar{\Phi}_{yb} V, \quad \mathcal{Z} = \bar{\Phi}_z X_0 + \bar{\Phi}_{zb} V \tag{11}$$

Denoting the tangent of the measurements of the δ's as

$$m_i = \tan\delta_i = z_i/x_i \quad (i = 1,2,3) \tag{12}$$

one can write

$$\mathcal{Z} = M\mathfrak{X} \quad M \equiv \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{13}$$

From the geometry of Fig. 1,

$$\tan\sigma = y/(x^2 + z^2)^{1/2} \text{ and } \sin\delta = z/(x^2 + z^2)^{1/2} \tag{14}$$

Hence

$$y_i = z_i \tan\sigma_i/\sin\delta_i \equiv q_i z_i \quad (i = 1,2,3) \tag{15}$$

and one can write

$$\mathcal{Y} = Q\mathcal{Z} \quad Q \equiv \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix} \tag{16}$$

Inserting Eq. (10) and $\mathcal{Z}$ from Eq. (11) into Eq. (13),

$$\bar{\Phi}_z X_0 + \bar{\Phi}_{zb} V = M\bar{\Phi}_x X_0 + M\bar{\Phi}_{xb} V \tag{17}$$

Transposing,

$$(\bar{\Phi}_z - M\bar{\Phi}_x)X_0 = (M\bar{\Phi}_{xb} - \bar{\Phi}_{zb})V \tag{18}$$

Similarly, inserting $\mathcal{Y}$ and $\mathcal{Z}$ from Eq. (11) into Eq. (16),

$$(\bar{\Phi}_y - Q\bar{\Phi}_z)X_0 = (Q\bar{\Phi}_{zb} - \bar{\Phi}_{yb})V \tag{19}$$

The 3×1 matrix equations in Eqs. (18) and (19) can be considered as partitions of the 6×1 equation:

$$AX_0 = BV \tag{20}$$

where

$$A \equiv \begin{bmatrix} \bar{\Phi}_z - M\bar{\Phi}_x \\ \hline \bar{\Phi}_y - Q\bar{\Phi}_z \end{bmatrix} \quad B \equiv \begin{bmatrix} M\bar{\Phi}_{xb} - \bar{\Phi}_{zb} \\ \hline Q\bar{\Phi}_{zb} - \bar{\Phi}_{yb} \end{bmatrix} \tag{21}$$

The solution of Eq. (20) yields the desired result,

$$X_0 = A^{-1}BV \tag{22}$$

Equation (22) is the only equation that need be solved by the computer, given the definitions of A, B, V, and the data. Once X_0 is known, state at any other time can be found using Eq. (4).

Using the subscript p to distinguish the four-instant in-plane solution, define the following symbols:

$$X_p \equiv [xzx'z']^T \tag{23}$$

$$\mathfrak{X}_p \equiv [x_1x_2x_3x_4]^T \tag{24}$$

$$\mathcal{Z}_p \equiv [z_1z_2z_3z_4]^T \tag{25}$$

$$M_p \equiv \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \tag{26}$$

$$V_p \equiv [00\Delta x'\Delta z']^T \tag{27}$$

$$\Psi_{ib} \equiv \begin{cases} [0] & \text{if } \tau_i < \tau_b \\ \Psi(\tau_i,\tau_b) & \text{if } \tau_i \geq \tau_b \end{cases} \tag{28}$$

The in-plane state X_{ip} at any time τ_i can be expressed in terms of initial conditions as follows:

$$X_{ip} = \Psi_{i0}X_{0p} + \Psi_{ib}V_p \quad (i = 1,2,3,4) \tag{29}$$

The individual components of position at τ_i are

$$x_i = \Psi_{xi0}X_{0p} + \Psi_{xib}V_p \quad (i = 1,2,3,4) \tag{30}$$

$$z_i = \Psi_{zi0}X_{0p} + \Psi_{zib}V_p \quad (i = 1,2,3,4) \tag{31}$$

where Ψ_{xi0} and Ψ_{zi0} are the first and second rows respectively of Ψ_{i0}. Combining all four values of x into a matrix,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \Psi_{x10} \\ \Psi_{x20} \\ \Psi_{x30} \\ \Psi_{x40} \end{bmatrix} [X_{0p}] + \begin{bmatrix} \Psi_{x1b} \\ \Psi_{x2b} \\ \Psi_{x3b} \\ \Psi_{x4b} \end{bmatrix} [V_p] \tag{32}$$

which is abbreviated, using prior conventions, as

$$\mathfrak{X}_p = \bar{\Psi}_x X_{0p} + \bar{\Psi}_{xb} V_p \tag{33}$$

and a similar equation applies for the z-components. Using Eqs. (26) and (12), one can write

$$\mathcal{Z}_p = M_p\mathfrak{X}_p \tag{34}$$

Substituting $\mathcal{Z}_p$ and $\mathfrak{X}_p$, and transposing,

$$(\bar{\Psi}_z - M_p\bar{\Psi}_x)X_{0p} = (M_p\bar{\Psi}_{xb} - \bar{\Psi}_{zb})V_p \tag{35}$$

Solving for X_{0p} yields the results for the in-plane state

$$X_{0p} = (\bar{\Psi}_z - M_p\bar{\Psi}_x)^{-1}(M_p\bar{\Psi}_{xb} - \bar{\Psi}_{zb})V_p \tag{36}$$

Knowing the in-plane state at τ_0, the in-plane state at any other time can be found using Eq. (29).

The out-of-plane state can be found by obtaining the values y_j,y_k using Eq. (15) for any two measurement times between which there was no y component of thrust;

$$y_j = q_jz_j, \quad y_k = q_kz_k \tag{37}$$

By virtue of the nonthrust, it is clear from Eq. (3) that $y_k = c_{kj}y_j + s_{kj}y_j'$, which is solved for y_j'

$$y_j' = (y_k - c_{kj}y_j)/s_{kj} \tag{38}$$

Equations (36–38) are implemented in the computer, completing the determination of state at τ_j in the four-instant procedure. Knowing the complete state at τ_j, the state at any other time is found using Eq. (4), identifying 0 with j.

When an out-of-plane thrust maneuver is contemplated, the three-instant out-of-plane procedure should be used. When an in-plane thrust maneuver is contemplated, the four-instant in-plane procedure will ordinarily give the most accurate results. Whereas in principle the three-instant procedure can be used with any finite burn, in the case in which burn has no y component this procedure will give accurate results only when the out-of-plane distance y is comparable with the in-plane distance $(x^2 + z^2)^{1/2}$.

The 3-instant out-of-plane procedure using Eq. (22) requires inversion of a 6×6 matrix. The 4-instant in-plane procedure using Eq. (36) requires inversion of a 4×4 matrix. Obviously the solution can be implemented in other ways, since both were solved in the manual method of Ref. 1 by inverting nothing larger than 2×2 matrices.

There are occasions when one or the other of the above methods will fail, this being evidenced by the difficulty in inverting the matrix. Under these circumstances, the difficulty is usually corrected by using an extra measurement of δ and σ in place of one δ-σ pair from the original set.

The algorithms presented here can be extended to the case of moderately eccentric reference orbits using the techniques of Ref. 3, in which the x,y,z axes are simply reinterpreted as defined by a local tangent coordinate system, there being no change to the equations of motion or their solution.

The algorithms presented here can also be extended to the case where separation distances along the orbit are longer than are accurately described by the linear equations (1) by using the technique of Ref. 4. There, curvilinear axes are introduced in which the equations of motion and their solution again have exactly the same form as given here.

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Thrust-Minus-Drag Optimization by Base Bleed and/or Boattailing

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Nomenclature

$A, \bar{A}$	=	area; area ratio, A/A_{2E}, respectively
$\bar{B}_0$	=	base bleed-to-nozzle mass flow-rate ratio
C_D	=	afterbody-base drag coefficient = $C_{DB} + C_{DBT}$
C_{DBT}, C_{DB}	=	boattail and base drag coefficients, respectively
F_{NET}	=	thrust-minus-drag force
$\bar{L}$	=	boattail length-to-body radius ratio
M	=	Mach number
$P, \bar{P}$	=	absolute pressure; static-to-freestream pressure ratio, P/P_E, respectively
$\bar{\mathcal{R}}_I$	=	ratio of gas constants, $\mathcal{R}_I/\mathcal{R}_E$
V	=	velocity magnitude
X, R	=	longitudinal and radial coordinates, respectively
ρ	=	density
γ	=	ratio of specific heats
β	=	flow angle
ΔC_F	=	incremental thrust-minus-drag coefficient, Eq. (6)

Subscripts

B,BT	=	base and boattail regions, respectively
E,I	=	external (freestream) and internal (nozzle) flows, respectively
R	=	reference configuration
X	=	component in the longitudinal direction
0	=	stagnation conditions
1	=	geometric separation point at terminus of nozzle or afterbody
2	=	initial afterbody point

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Introduction

DURING powered supersonic flight, base drag due to the interaction between the propulsive-nozzle and freestream flows is a significant part of the over-all drag of a vehicle.[1,2] Two effective techniques for reducing base drag are mass bleed into the base region and/or using a boattailed afterbody. Evaluation of the effects of these techniques on vehicle performance must also consider the tradeoffs involved. For example, when the base-bleed flow is diverted from the propulsive-nozzle stagnation chamber, the gain achieved by base-drag reduction must be considered relative to the thrust that would have been produced if the base-bleed flow had been expanded through the propulsive nozzle. When a boattail is used, the tradeoff between afterbody drag and base drag must be considered. If both techniques are involved, the individual effects and their interaction must be evaluated. As a consequence, the evaluation and possible optimization procedures can be judged best on a unified basis by the relative gain achieved in the thrust-minus-drag force, F_{NET}.[3] Presented herein are the bases for such evaluations and example results for cylindrical and conical afterbodies (Fig. 1). The flowfield over the afterbody is determined by the Method of Characteristics and the base-flow analysis is based on the flow model of Korst, et al.[4] These analyses have been incorporated into computer programs currently available.[5,6]

Analysis

For the control volume of Fig. 1b

$$F_{NET} = \rho_{1I}A_{1I}V_{1I}^2 - \int_{(A_{BT})_X} (P_E - P_{BT})\, d(A_{BT})_X - (P_E - P_B)(A_{1E} - A_{1I}) - (P_E - P_{1I})A_{1I} \quad (1)$$

In Eq. (1), the effects of friction have been neglected and the base-bleed flow, if any, is assumed to possess negligible momentum. Using freestream conditions and the maximum body cross-sectional area as reference quantities, Eq. (1) can be expressed in nondimensional form as

$$F_{NET}/(\rho_E V_E^2 A_{2E}/2) = -C_{DBT} - C_{DB} - [\bar{A}_{1I} - \bar{P}_{1I}\bar{A}_{1I}(1 + \gamma_I M_{1I}^2)]/(\gamma_E M_E^2/2) \quad (2)$$

The boattail and base drag coefficients are defined, respectively, as

$$C_{DBT} = \int_{(\bar{A}_{BT})_X} (1 - \bar{P}_{BT})\, d(\bar{A}_{BT})_X/(\gamma_E M_E^2/2) \quad (3)$$

and

$$C_{DB} = (1 - \bar{P}_B)(\bar{A}_{1E} - \bar{A}_{1I})/(\gamma_E M_E^2/2) \quad (4)$$

The over-all afterbody-base drag coefficient is

$$C_D = C_{DBT} + C_{DB} \quad (5)$$

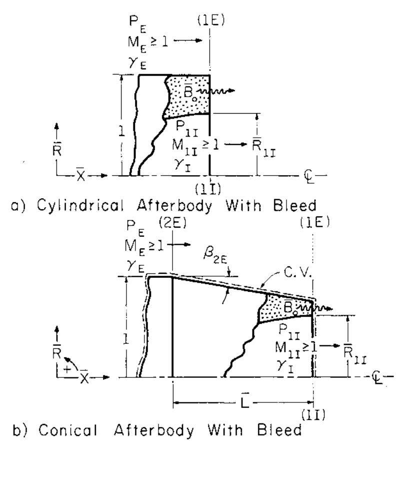

Fig. 1 Configurations and notation.